

# Stochastic Calculus for Assets with Non-Gaussian Price Fluctuations \*

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From the path integral formalism for price fluctuations with non-Gaussian distributions I derive the appropriate stochastic calculus replacing Itô's calculus for stochastic fluctuations.

## I. INTRODUCTION

The logarithms of assets prices in financial markets do not fluctuate with Gaussian distributions. This was first noted by Pareto in the 19th century [1] and emphasized by Mandelbrot in the 1960s [2]. The true distributions possess large tails which may be approximated by various other distributions, most prominently the truncated Lévy distributions [2–5], the Meixner distributions [6,7], or the generalized hyperbolic distributions and their descendents [8–32]. The stochastic differential equations associated with such distributions cannot be treated with the popular Itô calculus. The purpose of this paper is to develop an appropriate calculus to replace it. In Section II we briefly recapitulate the Gaussian approximation and set the stage for the generalization in Section III.

## II. GAUSSIAN APPROXIMATION TO FLUCTUATION PROPERTIES OF STOCK PRICES

Let  $S(t)$  denote the price of some stock. Over long time spans, the average over many stock prices has a time behavior which can be approximated by pieces of exponentials. This is why they are usually plotted on a logarithmic scale. For an illustration see the Dow-Jones industrial index over 60 years in Fig. 1.

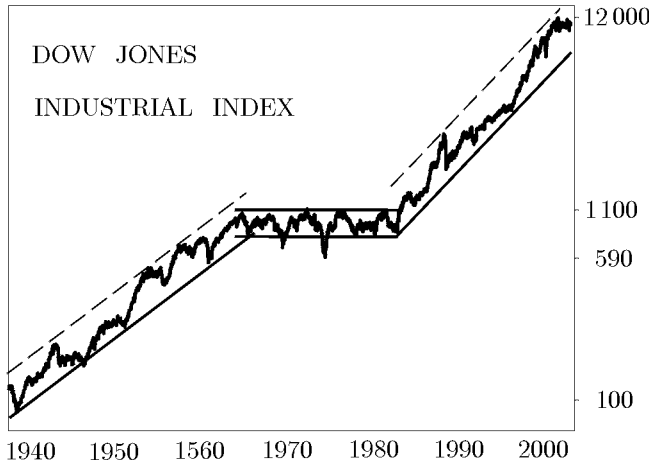


FIG. 1. Periods of exponential growth of price index averaged over major industrial stocks in the United States over 60 years.

For a liquid market with many participants, the price fluctuations seem to be driven by a stochastic noise with a white spectrum, as illustrated in Fig. 2.

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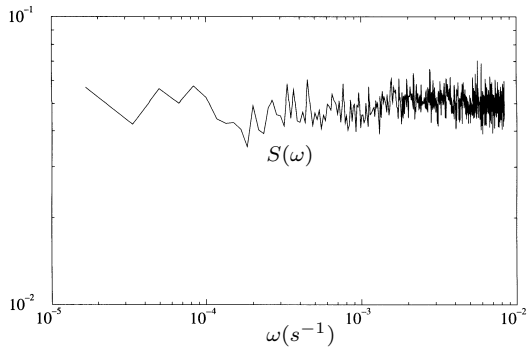


FIG. 2. Fluctuation spectrum of exchange rate DM/US\$ as a function of the frequency, indicating a white noise in the stochastic differential equation (1) (from the textbook [5]).

The fluctuations of the index have a certain average width called the *volatility* of the market. Over longer time spans, the volatility changes stochastically, as illustrated by the data of the S&P500 index over the years 1984-1997 shown in Fig. 3. In particular, there are strong increases short before market

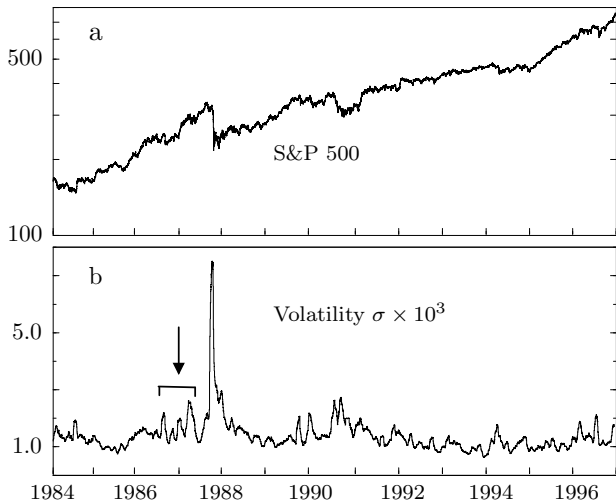


FIG. 3. (a) The S&P 500 index for the 13-year period 1 Jan 1984 - 31 Dec 1996 at interval of 1 min. Large fluctuations appeared on 19 Oct 1987 (*black Monday*). (b) Volatility over times  $T=1$  mon (8190 min) for courses taken every 30min. The precursors of the '87 crash are indicated by arrows (from Ref. [33]).

crashes. The theory to be developed here will ignore these fluctuations and assume a constant volatility. For recent work taking them into account see [34].

The distribution of the logarithms of the volatilities is normal as shown in Fig. 4.

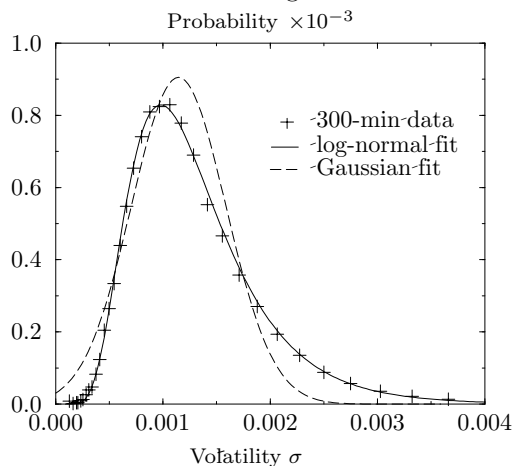


FIG. 4. Comparison of the best log-normal and Gaussian fits to volatilities over 300-min (from Ref. [33]).

An individual stock will in general have larger volatility than an average market index, especially when the company is small and only few stocks are traded per day.

To lowest approximation, the stock price  $S(t)$  satisfies the simplest stochastic differential equation for exponential growth

$$\frac{\dot{S}(t)}{S(t)} = r_S + \eta(t), \quad (1)$$

where  $\eta(t)$  is a white noise with the correlation functions

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t) \eta(t') \rangle = \sigma^2 \delta(t - t'). \quad (2)$$

The standard deviation  $\sigma$  parametrizes the volatility of the stock price, which is measured by the expectation value

$$\left\langle \left[ \frac{\dot{S}(t)}{S(t)} \right]^2 \right\rangle dt = \sigma^2. \quad (3)$$

The logarithm of the stock price

$$x(t) = \log S(t) \quad (4)$$

does not simply satisfy the stochastic linear-growth differential equation  $\dot{x}(t) = \dot{S}(t)/S(t) = r_S + \eta(t)$ , but possesses another growth rate  $r_x$ :

$$\dot{x}(t) = r_x + \eta(t). \quad (5)$$

The change of the growth rate is due the stochastic nature of  $x(t)$  and  $S(t)$ . According to Itô's rule, a function of a stochastic variable satisfies the relation

$$\dot{f}(x(t)) = \partial_x f(x(t)) \dot{x}(t) + \frac{\sigma^2}{2} f''(x(t)). \quad (6)$$

This is derived by expanding  $f(x(t + \epsilon))$  in a Taylor series

$$\begin{aligned} f(x(t + \epsilon)) &= f(x(t)) + f'(x(t)) \int_t^{t+\epsilon} dt' \dot{x}(t') \\ &+ \frac{1}{2} f''(x(t)) \int_t^{t+\epsilon} dt_1 \int_t^{t+\epsilon} dt_2 \dot{x}(t_1) \dot{x}(t_2) \\ &+ \frac{1}{3!} f^{(3)}(x(t)) \int_t^{t+\epsilon} dt_1 \int_t^{t+\epsilon} dt_2 \int_t^{t+\epsilon} dt_3 \dot{x}(t_1) \dot{x}(t_2) \dot{x}(t_3) + \dots, \end{aligned} \quad (7)$$

replacing the higher correlation functions on the right-hand side by their expectation values, and observing that only the quadratic noise term in  $\dot{x}(t) = r_x + \eta(t)$  contributes to linear order in  $\epsilon$ . For the logarithmic function (4), the expansion yields

$$\begin{aligned} \dot{x}(t) &= \frac{dx}{dS} \dot{S}(t) + \frac{1}{2} \frac{d^2 x}{dS^2} \dot{S}^2(t) dt + \dots \\ &= \frac{\dot{S}(t)}{S(t)} - \frac{1}{2} \left[ \frac{\dot{S}(t)}{S(t)} \right]^2 dt + \dots, \end{aligned} \quad (8)$$

which becomes, after the replacement of the quadratic term by its expectation value (3),

$$\dot{x}(t) = \frac{\dot{S}(t)}{S(t)} - \frac{1}{2} \sigma^2 + \dots \quad (9)$$

Inserting here Eqs. (1), and (5), we find that the linear growth rate of  $x(t)$  is related to the exponential growth rate of  $S(t)$  by

$$r_x = r_S - \frac{1}{2}\sigma^2. \quad (10)$$

In praxis, this relation implies that if we fit a straight line through a plot of the logarithms of stock prices, the forward extrapolation of the average stock price is given by

$$\langle S(t) \rangle = S(0) e^{r_S t} = S(0) e^{(r_x + \sigma^2/2)t}. \quad (11)$$

A typical set of solutions of the stochastic differential equation (5) is shown in Fig. 5.

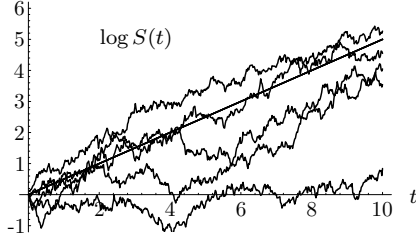


FIG. 5. Behavior of logarithm of stock price following the stochastic differential equation (1).

### III. LÉVY DISTRIBUTIONS

The description of the fluctuations of the logarithms of the stock prices around the linear trend by a Gaussian distribution is only a rough approximation to the real stock prices. As explained before, these have volatilities depending on time. More severely, if observed in smaller time intervals, for instance every day or hour, they have distributions in which rare events have a much higher probability than in Gaussian distributions, whose exponential tails are extremely small. Following Pareto [1], Mandelbrot emphasized in the 1960s that fluctuations of financial assets could be fitted much better with the help of Lévy distributions [2]. These distributions are defined by the Fourier transform

$$\tilde{L}_{\sigma^2}^{\mu}(x) \equiv \int_{-\infty}^{\infty} \frac{dp}{2\pi} L_{\sigma^2}^{\mu}(p) e^{ipx}, \quad (12)$$

with

$$L_{\sigma^2}^{\mu}(p) \equiv \exp \left[ -(\sigma^2 p^2)^{\mu/2} / 2 \right]. \quad (13)$$

The Gaussian distribution are recovered in the limit  $\mu \rightarrow 2$ .

For large  $x$ , the Lévy distribution (12) falls off with the characteristic power behavior

$$\tilde{L}_{\sigma^2}^{\mu}(x) \rightarrow A_{\sigma^2}^{\mu} \frac{\mu}{|x|^{1+\mu}}. \quad (14)$$

These power falloffs are referred to as *Paretian tails* of the distributions. The amplitude of the tails is found by approximating the integral (12) for large  $x$ , where only small momenta contribute, as follows

$$\tilde{L}_{\sigma^2}^{\mu}(x) \approx \int_{-\infty}^{\infty} \frac{dp}{2\pi} \left[ 1 - \frac{1}{2}(\sigma^2 p^2)^{\mu/2} \right] e^{ipx} \xrightarrow{x \rightarrow \infty} A_{\sigma^2}^{\mu} \frac{\mu}{|x|^{1+\mu}}, \quad (15)$$

with

$$A_{\sigma^2}^{\mu} = -\frac{\sigma^{\mu}}{2\mu} \int_0^{\infty} \frac{dp'}{\pi} p'^{\mu} \cos p' = \frac{\sigma^{\mu}}{2\pi\mu} \sin(\pi\mu/2) \Gamma(1+\mu). \quad (16)$$

The stock market data are fitted best with  $\mu$  between 1.2 and 1.5 [5], and we shall use  $\mu = 3/2$  most of the time for simplicity, where one has

$$A_{\sigma^2}^{3/2} = \frac{1}{4} \frac{\sigma^{3/2}}{\sqrt{2\pi}}. \quad (17)$$

The full Taylor expansion of the Fourier transform (13) yields the asymptotic series

$$\tilde{L}_{\sigma^2}^{\mu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{\infty} \frac{dp}{\pi} \frac{\sigma^{\mu n} p^{\mu n}}{2^n} \cos px = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{\sigma^{\mu n}}{2^n \pi} \Gamma(1+n\mu) \sin(\pi\mu/2) \frac{1}{|x|^{1+\mu}}. \quad (18)$$

This series is not useful for practical calculations since it fails to reproduce the exponential tails of the distribution, a typical shortcoming of large- $x$  expansions. In particular, it does not reduce to the Gaussian distribution in the limit  $\mu \rightarrow 2$ .

### A. Truncated Lévy Distributions

An undesirable property of the Lévy distributions which is incompatible with financial data is that their fluctuation width diverges for  $\mu < 2$ , since

$$\sigma^2 = \langle x^2 \rangle \equiv \int_{-\infty}^{\infty} dx x^2 \tilde{L}_{\sigma^2}^{\mu}(x) = -\frac{d^2}{dp^2} L_{\sigma^2}^{\mu}(p) \Big|_{p=0} \quad (19)$$

is infinite. In contrast, real stock prices have a finite width. To account for both Paretian tails *and* finite width one may introduce the so-called *truncated Lévy distributions* [3]. They are defined by

$$\tilde{L}_{\sigma^2}^{(\mu, \alpha)}(x) \equiv \int_{-\infty}^{\infty} \frac{dp}{2\pi} L_{\sigma^2}^{(\mu, \alpha)}(p) e^{ipx}, \quad (20)$$

with a Fourier transform  $L_{\sigma^2}^{(\mu, \alpha)}(p)$  which generalizes the function (13). It is conveniently written as an exponential of a “Hamiltonian function”  $H(p)$ :

$$L_{\sigma^2}^{(\mu, \alpha)}(p) \equiv e^{-H(p)}, \quad (21)$$

with

$$\begin{aligned} H(p) &\equiv \frac{\sigma^2}{2} \frac{\alpha^{2-\mu}}{\mu(1-\mu)} [(\alpha + ip)^{\mu} + (\alpha - ip)^{\mu} - 2\alpha^{\mu}] \\ &= \sigma^2 \frac{(\alpha^2 + p^2)^{\mu/2} \cos[\mu \arctan(p/\alpha)] - \alpha^{\mu}}{\alpha^{\mu-2} \mu(1-\mu)}. \end{aligned} \quad (22)$$

The asymptotic behavior of the truncated Lévy distributions differs from the power behavior of the Lévy distribution in Eq. (15) by an exponential factor  $e^{-\alpha x}$ , which guarantees the finiteness of the width  $\sigma$  and of all higher moments. An estimate of the leading term is again obtained from the Fourier transform

$$\begin{aligned} \tilde{L}_{\sigma^2}^{(\mu, \alpha)}(x) &\approx e^{2C\alpha^{\mu}} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \{1 + C[(\alpha + ip)^{\mu} + (\alpha - ip)^{\mu}]\} e^{ipx} \\ &\xrightarrow{x \rightarrow \infty} e^{2C\alpha^{\mu}} \Gamma(1+\mu) \frac{\sin(\pi\mu)}{\pi} C \frac{e^{-\alpha|x|}}{|x|^{1+\mu}}, \end{aligned} \quad (23)$$

where

$$C \equiv \frac{\sigma^2}{2} \frac{\alpha^{2-\mu}}{\mu(1-\mu)}. \quad (24)$$

The integral follows directly from the formulas [35]

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} (\alpha + ip)^{\mu} e^{ipx} = \frac{\Theta(x)}{\Gamma(-\mu)} \frac{e^{-\alpha x}}{x^{1+\mu}}, \quad \int_{-\infty}^{\infty} \frac{dp}{2\pi} (\alpha - ip)^{\mu} e^{ipx} = \frac{\Theta(-x)}{\Gamma(-\mu)} \frac{e^{-\alpha|x|}}{|x|^{1+\mu}}, \quad (25)$$

and the identity for Gamma functions  $1/\Gamma(-z) = -\Gamma(1+z) \sin(\pi z)/\pi$ . The full expansion is integrated with the help of the formula [36]

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{dp}{2\pi} (\alpha + ip)^\mu (\alpha - ip)^\nu e^{ipx} \\
&= (2\alpha)^{\mu/2+\nu/2} \frac{1}{|x|^{1+\mu/2+\nu/2}} \begin{cases} \frac{1}{\Gamma(-\mu)} W_{(\nu-\mu)/2, (1+\mu+\nu)/2}(2\alpha x) & x > 0, \\ \frac{1}{\Gamma(-\nu)} W_{(\mu-\nu)/2, (1+\mu+\nu)/2}(2\alpha x) & x < 0, \end{cases} \quad \text{for} \quad (26)
\end{aligned}$$

where the Whittaker functions  $W_{(\nu-\mu)/2, (1+\mu+\nu)/2}(2\alpha x)$  can be expressed in terms of Kummer's confluent hypergeometric function  ${}_1F_1(a; b; z)$  as

$$\begin{aligned}
W_{\lambda, \kappa}(z) &= \frac{\Gamma(-2\kappa)}{\Gamma(1/2 - \kappa - \lambda)} z^{\kappa+1/2} e^{-z/2} {}_1F_1(1/2 + \kappa - \lambda; 2\kappa + 1; z) \\
&+ \frac{\Gamma(2\kappa)}{\Gamma(1/2 + \kappa - \lambda)} z^{-\kappa+1/2} e^{-z/2} {}_1F_1(1/2 - \kappa - \lambda; -2\kappa + 1; z). \quad (27)
\end{aligned}$$

For  $\nu = 0$ , only  $x > 0$  gives a nonzero integral (26), which reduces, with  $W_{-\mu/2, 1/2+\mu/2}(z) = z^{-\mu/2} e^{-z/2}$ , to the left equation in (25). Setting  $\mu = \nu$  we find

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} (\alpha^2 + p^2)^\nu e^{ipx} = (2\alpha)^{\nu/2} \frac{1}{|x|^{1+\nu}} \frac{1}{\Gamma(-\nu)} W_{0, 1/2+\nu}(2\alpha|x|). \quad (28)$$

Inserting

$$W_{0, 1/2+\nu}(z) = \sqrt{\frac{2z}{\pi}} K_{1/2+\nu}(z/2), \quad (29)$$

we may write

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} (\alpha^2 + p^2)^\nu e^{ipx} = \left( \frac{2\alpha}{|x|} \right)^{1/2+\nu} \frac{1}{\sqrt{\pi} \Gamma(-\nu)} K_{1/2+\nu}(\alpha|x|). \quad (30)$$

For  $\nu = -1$  where  $K_{-1/2}(z) = K_{1/2}(z) = \sqrt{\pi/2z} e^{-z}$ , this reduces to

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{1}{\alpha^2 + p^2} e^{ipx} = \frac{1}{2\alpha} e^{-\alpha|x|}. \quad (31)$$

Summing up all terms in the expansion

$$\tilde{L}_{\sigma^2}^{(\mu, \alpha)}(x) \approx e^{2C\alpha^\mu} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-C)^n}{n!} [(\alpha + ip)^\mu + (\alpha - ip)^\mu]^n \right\} e^{ipx} \quad (32)$$

yields the true asymptotic behavior which differs from the estimate (23) by a constant factor [37]:

$$\tilde{L}_{\sigma^2}^{(\mu, \alpha)}(x) \xrightarrow{x \rightarrow \infty} e^{(2-2^\mu)C\alpha^\mu} \Gamma(1 + \mu) \frac{\sin(\pi\mu)}{\pi} C \frac{e^{-\alpha|x|}}{|x|^{1+\mu}}. \quad (33)$$

In contrast to Gaussian distributions which are characterized completely by the width  $\sigma$ , the truncated Lévy distributions contain three parameters  $\sigma$ ,  $\mu$ , and  $\alpha$ . Best fits to two types of fluctuating market prices are shown in Fig. 6, in which we plot the cumulative probabilities

$$P_{<}(\delta x) = \int_{-\infty}^{\delta x} dx \tilde{L}_{\sigma^2}^{(\mu, \alpha)}(x), \quad P_{>}(\delta x) = \int_{\delta x}^{\infty} dx \tilde{L}_{\sigma^2}^{(\mu, \alpha)}(x) = 1 - P_{<}(\delta x). \quad (34)$$

For negative price fluctuations  $\delta x$ , the plot shows  $P_{<}(\delta x)$ , for positive price fluctuations  $P_{>}(\delta x)$ . By definition,  $P_{<}(-\infty) = 0$ ,  $P_{<}(0) = 1/2$ ,  $P_{<}(\infty) = 1$ , and  $P_{>}(-\infty) = 1$ ,  $P_{>}(0) = 1/2$ ,  $P_{>}(\infty) = 0$ . To fit the general shape, one chooses an appropriate parameter  $\mu$  which turns out to be rather universal, close to  $\mu = 3/2$ . The remaining two parameters fix all expansion coefficients of Hamiltonian (22):

$$H(p) = \frac{1}{2}c_2 p^2 - \frac{1}{4!}c_4 p^4 + \frac{1}{6!}c_6 p^6 - \frac{1}{8!}c_8 p^8 + \dots \quad (35)$$

The numbers  $c_n$  are referred to as the *cumulants* of the truncated Lévy distribution. They are equal to

$$\begin{aligned} c_2 &= \sigma^2, \\ c_4 &= \sigma^2(2 - \mu)(3 - \mu)\alpha^{-2}, \\ c_6 &= \sigma^2(2 - \mu)(3 - \mu)(4 - \mu)(5 - \mu)\alpha^{-4}, \\ &\vdots \\ c_{2n} &= \sigma^2 \frac{\Gamma(2n - \mu)}{\Gamma(2 - \mu)} \alpha^{2-2n}, \end{aligned} \quad (36)$$

$P_{\leq}(x)$

$P_{\leq}(x)$

see extra fig6a.pdf

see extra fig6b.pdf

$x$

$x$

FIG. 6. Best fit of cumulative versions (34) of truncated Lévy distribution,  $P_{<}(\delta x)$  for negative price fluctuations  $\delta x$ , and  $P_{>}(\delta x)$  for positive price fluctuations to the instantaneous fluctuations of the S&P 500 (l.h.s) and the DM/US\$ exchange rate measured every fifteen minutes. The negative fluctuations lie on a slightly higher curve than the positive ones. The difference is often neglected. The parameters  $A$  and  $\alpha$  are the size and truncation parameters of the distribution. The best value of  $\mu$  is  $3/2$  (from [5]).

The first determines the quadratic fluctuation width

$$\langle x^2 \rangle \equiv \int_{-\infty}^{\infty} dx x^2 \tilde{L}_{\sigma^2}^{(\mu, \alpha)}(x) = -\frac{d^2}{dp^2} L_{\sigma^2}^{(\mu, \alpha)}(p) \Big|_{p=0} = c_2 = \sigma^2, \quad (37)$$

the second the fourth-order expectation

$$\langle x^4 \rangle \equiv \int_{-\infty}^{\infty} dx x^4 \tilde{L}_{\sigma^2}^{(\mu, \alpha)}(x) = \frac{d^4}{dp^4} L_{\sigma^2}^{(\mu, \alpha)}(p) \Big|_{p=0} = c_4 + 3c_2^2, \quad (38)$$

and so on:

$$\langle x^6 \rangle = c_6 + 15c_4c_2 + 15c_2^3, \quad \langle x^8 \rangle = c_8 + 28c_6c_2 + 35c_4^2 + 210c_4c_2^2 + 105c_2^4, \dots \quad (39)$$

In analyzing the data, one usually defines the so-called *kurtosis*, which is the normalized fourth-order cumulant

$$\kappa \equiv \bar{c}_4 \equiv \frac{c_4}{c_2^2} = \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2} - 3. \quad (40)$$

It depends on the parameters  $\sigma, \mu, \alpha$  as follows

$$\kappa = \frac{(2-\mu)(3-\mu)}{\sigma^2 \alpha^2}. \quad (41)$$

Given the volatility  $\sigma$  and the kurtosis  $\kappa$ , we extract the Lévy parameter  $\alpha$  from the equation

$$\alpha = \frac{1}{\sigma} \sqrt{\frac{(2-\mu)(3-\mu)}{\kappa}}. \quad (42)$$

In terms of  $\kappa$  and  $\sigma^2$ , the expansion coefficients are

$$\begin{aligned} \bar{c}_4 &= \kappa, \quad \bar{c}_6 = \kappa^2 \frac{(5-\mu)(4-\mu)}{(3-\mu)(2-\mu)}, \quad \bar{c}_8 = \kappa^2 \frac{(7-\mu)(6-\mu)(5-\mu)(4-\mu)}{(3-\mu)^2(2-\mu)^2}, \\ &\vdots \\ \bar{c}_n &= \kappa^{n/2-1} \frac{\Gamma(n-\mu)/\Gamma(4-\mu)}{(3-\mu)^{n/2-2}(2-\mu)^{n/2-2}}. \end{aligned} \quad (43)$$

For  $\mu = 3/2$ , the second equation in (42) becomes simply

$$\alpha = \frac{1}{2} \sqrt{\frac{3}{\sigma^2 \kappa}}, \quad (44)$$

and the coefficients (45):

$$\begin{aligned} \bar{c}_4 &= \kappa, \quad \bar{c}_6 = \frac{5 \cdot 7}{3} \kappa^2, \quad \bar{c}_8 = 5 \cdot 7 \cdot 11 \kappa^2, \\ &\vdots \\ \bar{c}_n &= \frac{\Gamma(n-3/2)/\Gamma(5/2)}{3^{n/2-2}/2^{n-4}} \kappa^{n/2-1}. \end{aligned} \quad (45)$$

At zero kurtosis, the truncated Lévy distribution reduces to a Gaussian distribution of width  $\sigma$ . The change in shape for a fixed width and increasing kurtosis is shown in Fig. 7.

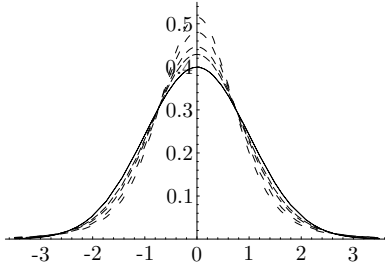


FIG. 7. Change in shape of truncated Lévy distributions of width  $\sigma = 1$  with increasing kurtoses  $\kappa = 0$  (Gaussian, solid curve), 1, 2, 5, 10.

From the S&P and DM/US\$ data with time intervals  $\Delta t = 15$  min one extracts  $\sigma^2 = 0.280$  and  $0.0163$ , and the kurtoses  $\kappa = 12.7$  and  $20.5$ , respectively. This implies  $M \approx 3.57$ ,  $\alpha \approx 0.46$  and  $M \approx 61.35$ ,  $\alpha \approx 1.50$ , respectively.

The other normalized cumulants ( $\bar{c}_6, \bar{c}_8, \dots$ ) are then all determined to be  $(1881.72, 788627.46, \dots)$  and  $(-4902.92, 3.3168 \times 10^6, \dots)$ , respectively. The cumulants increase rapidly showing that the expansion needs resummation.

From the data, the other normalized cumulants are found by evaluating the ratios of expectation values

$$\begin{aligned} \bar{c}_6 &= \frac{\langle x^6 \rangle}{\langle x^2 \rangle^3} - 15 \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2} + 30, \\ \bar{c}_8 &= \frac{\langle x^8 \rangle}{\langle x^2 \rangle^4} - 28 \frac{\langle x^6 \rangle}{\langle x^2 \rangle^3} - 35 \frac{\langle x^4 \rangle^2}{\langle x^2 \rangle^4} + 420 \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2} - 630, \dots \end{aligned} \quad (46)$$



## B. Asymmetric Truncated Lévy Distributions

We have seen in the data of Fig. 6 that the price fluctuations have a slight asymmetry: Price drops are slightly larger than rises. This is accounted for by an asymmetric truncated Lévy distribution. It has the general form [4]

$$L_{\sigma^2}^{(\lambda, \alpha, \beta)}(p) \equiv e^{-H(p)}, \quad (47)$$

with a Hamiltonian function

$$\begin{aligned} H(p) &\equiv \frac{\sigma^2}{2} \frac{\alpha^{2-\lambda}}{\lambda(1-\lambda)} [(\alpha + ip)^\lambda(1 + \beta) + (\alpha - ip)^\lambda(1 - \beta) - 2\alpha^\lambda] \\ &= \sigma^2 \frac{(\alpha^2 + p^2)^{\lambda/2} \{\cos[\lambda \arctan(p/\alpha)] + i\beta \sin[\lambda \arctan(p/\alpha)]\} - \alpha^\lambda}{\alpha^{\lambda-2}\lambda(1-\lambda)}. \end{aligned} \quad (48)$$

This has a power series expansion

$$H(p) = ic_1 p + \frac{1}{2}c_2 p^2 - i\frac{1}{3!}c_3 p^3 - \frac{1}{4!}c_4 p^4 + i\frac{1}{5!}c_5 p^5 + \dots, \quad (49)$$

which differs from (35) by the extra odd coefficients:

$$c_1 = \sigma^2 \frac{\alpha}{(1-\lambda)} \beta, \quad (50)$$

$$c_3 = \sigma^2 (2-\lambda) \alpha^{-1} \beta,$$

$$c_5 = \sigma^2 (2-\lambda)(3-\lambda)(4-\lambda) \alpha^{-3} \beta,$$

$$\vdots,$$

$$c_{2n+1} = \sigma^2 \frac{\Gamma(2n+1-\lambda)}{\Gamma(2-\lambda)} \alpha^{1-2n} \beta, \quad (51)$$

The general formula valid for even and odd  $n$  is

$$c_n = \sigma^2 \frac{\Gamma(n-\lambda)}{\Gamma(2-\lambda)} \alpha^{2-n} \begin{cases} 1 & n = \text{even}, \\ \beta & n = \text{odd}. \end{cases} \quad (52)$$

In addition to the even expectation values (37)–(39) there are now also odd expectation values:

$$\begin{aligned} \langle x \rangle &\equiv \int_{-\infty}^{\infty} dx x \tilde{L}_{\sigma^2}^{(\lambda, \alpha, \beta)}(x) = i \frac{d}{dp} e^{-H(p)} \Big|_{p=0} = c_1, \\ \langle x^2 \rangle &\equiv \int_{-\infty}^{\infty} dx x^2 \tilde{L}_{\sigma^2}^{(\lambda, \alpha, \beta)}(x) = -\frac{d^2}{d^2 p} e^{-H(p)} \Big|_{p=0} = c_2 + c_1^2, \\ \langle x^3 \rangle &\equiv \int_{-\infty}^{\infty} dx x^3 \tilde{L}_{\sigma^2}^{(\lambda, \alpha, \beta)}(x) = -i \frac{d^3}{d^3 p} e^{-H(p)} \Big|_{p=0} = c_3 + 3c_2 c_1 + c_1^3, \\ \langle x^4 \rangle &\equiv \int_{-\infty}^{\infty} dx x^4 \tilde{L}_{\sigma^2}^{(\lambda, \alpha, \beta)}(x) = \frac{d^4}{d^4 p} e^{-H(p)} \Big|_{p=0} = c_4 + 4c_3 c_1 + 3c_2^2 + 6c_2 c_1^2 + c_1^4, \\ &\vdots \end{aligned} \quad (53)$$

The distribution is now centered around a nonzero average value:

$$\mu \equiv \langle x \rangle = c_1. \quad (54)$$

The fluctuation width is given by

$$\sigma^2 \equiv \langle x^2 \rangle - \langle x \rangle^2 = \left\langle (x - \langle x \rangle)^2 \right\rangle = c_2. \quad (55)$$

The asymptotic behavior of the asymmetric truncated Lévy distributions is given by a straightforward modification of (25):

$$\begin{aligned}\tilde{L}_{\sigma^2}^{(\lambda, \alpha)}(x) &\approx \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx} \left\{ 1 - \frac{\sigma^2}{2} \frac{\alpha^{2-\lambda}}{\lambda(1-\lambda)} [(\alpha + ip)^\lambda(1+\beta) + (\alpha - ip)^\lambda(1-\beta) - 2\alpha^\lambda] \right\} \\ &\xrightarrow{x \rightarrow \infty} \sigma^2 e^{2s\alpha^\lambda} \Gamma(1+\lambda) \frac{\sin(\pi\lambda)}{\pi} s \frac{e^{-\alpha|x|}}{|x|^{1+\lambda}} [1 + \beta \operatorname{sgn}(x)].\end{aligned}\quad (56)$$

In analyzing the data, one now uses the *skewness*

$$\begin{aligned}s \equiv \bar{c}_3 &= \frac{c_3}{c_2^{3/2}} = \frac{1}{\sigma^3} (\langle x^3 \rangle - 3c_2c_1 - c_1^3) = \frac{1}{\sigma^3} \left( \langle x^3 \rangle - 3 \langle (x - \langle x \rangle)^2 \rangle \langle x \rangle - \langle x \rangle^3 \right) \\ &= \frac{1}{\sigma^3} \langle (x - \langle x \rangle)^3 \rangle.\end{aligned}\quad (57)$$

The kurtosis reads now

$$\begin{aligned}\kappa \equiv \bar{c}_4 &\equiv \frac{c_4}{c_2^2} = \frac{1}{\sigma^4} (\langle x^4 \rangle - 4c_3c_1 - 3c_2^2 - 6c_2c_1^2 - c_1^4) \\ &= \frac{1}{\sigma^4} \left[ \langle x^4 \rangle - 4 \langle (x - \langle x \rangle)^3 \rangle \langle x \rangle - 3 (\langle x^2 \rangle - \langle x \rangle^2)^2 - 6 (\langle x^2 \rangle - \langle x \rangle^2) \langle x \rangle^2 - \langle x \rangle^4 \right] \\ &= \frac{1}{\sigma^4} \left[ \langle (x - \langle x \rangle)^4 \rangle - 3 \langle x^2 - \langle x \rangle^2 \rangle^2 \right].\end{aligned}\quad (58)$$

It depends on the parameters  $\sigma$ ,  $\lambda$ ,  $\beta$ , and  $\alpha$  or  $\kappa$  as follows

$$s = \frac{(2-\lambda)\beta}{\sigma\alpha}.\quad (59)$$

The kurtosis has still the same dependence (41) on  $\sigma$ ,  $\lambda$ ,  $\alpha$  as before, such that we may also write

$$s = \sqrt{\frac{(2-\lambda)\kappa}{(3-\lambda)}} \beta.\quad (60)$$

The average position  $\mu = \langle x \rangle$  is proportional to the ratio of skewness and kurtosis:

$$\langle x \rangle = c_1 = \sigma^2 \frac{\alpha}{(1-\lambda)} \beta = \sigma \frac{3-\lambda}{1-\lambda} \frac{s}{\kappa}.\quad (61)$$

From the data one extracts volatility  $\sigma$ , kurtosis  $\kappa$ , and skewness  $s$  which determine completely the asymmetric truncated Lévy distribution. The data are then plotted against  $x - \langle x \rangle = x - \mu$ , so that they are centered at the average position. This centered distribution will be denoted by  $\bar{L}_{\sigma^2}^{(\lambda, \alpha, \beta)}(x)$ , i.e.

$$\bar{L}_{\sigma^2}^{(\lambda, \alpha, \beta)}(x) \equiv \tilde{L}_{\sigma^2}^{(\lambda, \alpha, \beta)}(x - \mu)\quad (62)$$

The Hamiltonian associated with this zero-average distribution is

$$\bar{H}(p) \equiv \bar{H}(p) - H'(0)p,\quad (63)$$

and its expansion in power of the momenta starts out with  $p^2$ , i.e., the first term in (49) is subtracted.

In terms of  $\kappa$ ,  $\sigma$ , and  $s$ , the normalized expansion coefficients are

$$\bar{c}_n = \kappa^{n/2-1} \frac{\Gamma(n-\lambda)/\Gamma(4-\lambda)}{(3-\lambda)^{n/2-2}(2-\lambda)^{n/2-2}} \begin{cases} 1 & n = \text{even}, \\ \sqrt{(3-\lambda)/(2-\lambda)\kappa} s & n = \text{odd}. \end{cases} \quad (64)$$

The change in shape of the distributions of a fixed width and kurtosis with increasing skewness is shown in Fig. 8. We have plotted the distributions centered around the average position  $x = \langle c_1 \rangle$  which means

that we have removed from  $H(p)$  in (47), (48), and (49) the linear term  $ic_1p$ . This subtracted Hamiltonian whose power series expansion begins with the term  $c_2p^2/2$  will be denoted by

$$\bar{H}(p) \equiv H(p) - H'(0) = \frac{1}{2}c_2p^2 - i\frac{1}{3!}c_3p^3 - \frac{1}{4!}c_4p^4 + i\frac{1}{5!}c_5p^5 + \dots \quad (65)$$

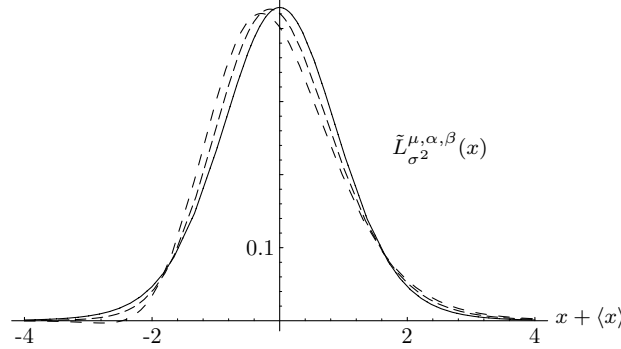


FIG. 8. Change in shape of truncated Lévy distributions of width  $\sigma = 1$  and kurtosis  $\kappa = 1$  with increasing skewness  $s = 0$  (solid curve), 0.4, 0.8. The curves are centered around  $\langle x \rangle$ .

### C. Meixner Distributions

Quite reasonable fits to financial data are provided by the *Meixner distributions* [6,7]. They have the advantage that they can be given explicitly in configuration and momentum space

$$\tilde{M}(x) = \frac{[2 \cos(b/2)]^{2d}}{2a\pi\Gamma(2d)} |\Gamma(d + ix/a)|^2 \exp[bx/a], \quad (66)$$

$$M(p) = \left\{ \frac{\cos(b/2)}{\cosh[(ap - ib)/2]} \right\}^{2d}. \quad (67)$$

They have the same tail behavior as the truncated Lévy distributions

$$\tilde{M}(x) \rightarrow C_{\pm} |x|^{\rho} e^{-\sigma_{\pm}|x|} \quad \text{for } x \rightarrow \pm\infty, \quad (68)$$

with

$$C_{\pm} = \frac{[2 \cos(b/2)]^{2d}}{2a\pi\Gamma(2d)} \frac{2\pi}{a^{2d-1}} e^{\pm 2\pi d \tan(b/2)}, \quad \rho = 2d - 1, \quad \sigma_{\pm} \equiv (\pi \pm b)/a. \quad (69)$$

The moments are

$$\begin{aligned} \mu &= ad \tan(b/2), & \sigma^2 &= a^2 d / 2 \cos^2(b/2), \\ s &= -\sqrt{2} \sin(b/2) / \sqrt{d}, & \kappa &= [2 - \cos b] / d, \end{aligned} \quad (70)$$

such that we can calculate the parameters from the moments as follows:

$$a^2 = \sigma^2 (2\kappa - 3s^2), \quad d = \frac{1}{\kappa - s^2}, \quad b = -2 \arcsin \left( s \sqrt{d/2} \right). \quad (71)$$

The Meixner distributions can be fitted quite well to the truncated Lévy distribution in the regime of large probability. In doing so we observe that the variance  $\sigma^2$  and the kurtosis  $\kappa$  are not the best parameters to match the two distributions. The large-probability regime of the distributions can be matched perfectly by choosing, in the symmetric case, the value and the curvature at the origin to be the same in both curves. This is seen in Fig. 9.

In the asymmetric case we have to match also the first and third derivatives. The derivatives of the Meixner distribution are:

$$\begin{aligned}
\tilde{M}(0) &= \frac{2^{2d-1}\Gamma^2(d)}{\pi a\Gamma(2d)}, \\
\tilde{M}'(0) &= b \frac{2^{2d-1}\Gamma^2(d) [1 - d\psi(d)]}{\pi a^2\Gamma(2d)}, \\
\tilde{M}''(0) &= -\frac{2^{2d}\Gamma^2(d)\psi(d)}{\pi a^3\Gamma(2d)}, \\
\tilde{M}^{(3)}(0) &= -\frac{b}{2} \frac{2^{2d}\Gamma^2(d) [6\psi(d) - 6d\psi^2(d) - d\psi^{(3)}(d)]}{\pi a^4\Gamma(2d)}, \\
\tilde{M}^{(4)}(0) &= \frac{2^{2d}\Gamma^2(d) [6\psi^2(d) + \psi^{(3)}(d)]}{\pi a^5\Gamma(2d)},
\end{aligned} \tag{72}$$

where  $\psi^{(n)}(z) \equiv d^{n+1} \log \Gamma(z)/dz^{n+1}$  are the Polygamma functions.

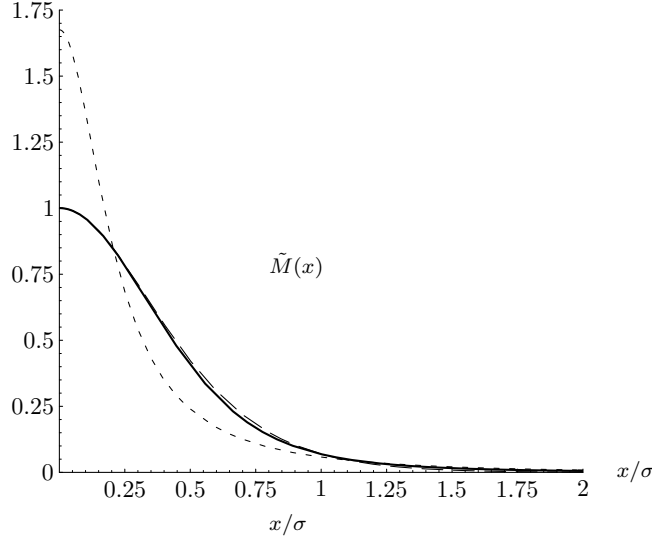


FIG. 9. Comparison of best fit of Meixner distribution to truncated Lévy distribution one by using the same  $\sigma^2$  and the kurtosis  $\kappa$  (short dashed) and once using the same value and curvature at the origin (long-dashed). The parameters are  $\sigma^2 = 0.280$  and  $\kappa = 12.7$  as in the left-hand cumulative distribution in Fig. 6. The Meixner distribution with the same  $\sigma^2$  and  $\kappa$  has parameters  $a = 2.666$ ,  $d = 0.079$ ,  $b = 0$ , the distribution with the same value and curvature at the origin has  $a = 0.6145$ ,  $d = 1.059$ ,  $b = 0$ . The very large  $\sigma$ -regime, however, is not fitted well as can be seen in the cumulative distributions which reach out to  $x$  of the order of  $10\sigma$  in Fig. 6.

#### D. Other Non-Gaussian Distributions

As mentioned in the beginning, some processes are also fitted quite well by generalized hyperbolic distributions [8–32]. They read

$$\tilde{H}_G(x) = \frac{(\gamma^2 - \beta^2)^{\lambda/2} e^{\beta x}}{\gamma^{\lambda-1/2} \delta^\lambda \sqrt{2\pi}} [\delta^2 + x^2]^{\lambda/2-1/4} \frac{K_{\lambda-1/2}(\gamma \sqrt{\delta^2 + x^2})}{K_\lambda(\delta \sqrt{\gamma^2 - \beta^2})} \tag{73}$$

and

$$G(p) = \frac{\delta^\lambda (\gamma^2 - \beta^2)^{\lambda/2}}{K_\lambda(\delta \sqrt{\gamma^2 - \beta^2})} \frac{K_\lambda(\delta \sqrt{\gamma^2 - (\beta + ip)^2})}{[\delta \sqrt{\gamma^2 - (\beta + ip)^2}]^\lambda}, \tag{74}$$

the latter defining another Hamiltonian

$$H_G(p) \equiv -\log G(p). \quad (75)$$

Introducing the variable  $\zeta \equiv \delta\sqrt{\gamma^2 - \beta^2}$ , this can be expanded in powers of  $p$  as in Eq. (35), yielding the first two cumulants:

$$c_1 = \beta \frac{\delta^2}{\zeta} \frac{K_{1+\lambda}(\zeta)}{K_\lambda(\zeta)}; \quad (76)$$

$$c_2 = \frac{\delta^2}{\zeta} \frac{K_{1+\lambda}(\zeta)}{K_\lambda(\zeta)} + \frac{\beta^2 \delta^4}{\zeta^2} \left\{ \frac{K_{2+\lambda}(\zeta)}{K_\lambda(\zeta)} - \left[ \frac{K_{1+\lambda}(\zeta)}{K_\lambda(\zeta)} \right]^2 \right\}. \quad (77)$$

Using the identity

$$K_{\nu+1}(z) - K_{\nu-1}(z) = \frac{2\nu}{z} K_\nu(z), \quad (78)$$

the latter equation can be expressed entirely in terms of

$$r = r(\zeta) = \frac{K_{1+\lambda}(\zeta)}{K_\lambda(\zeta)} \quad (79)$$

as

$$\begin{aligned} c_2 &= \frac{\delta^2}{\zeta} \frac{K_{1+\lambda}(\zeta)}{K_\lambda(\zeta)} + \frac{\beta^2 \delta^4}{\zeta^3} \left\{ \zeta + 2(1+\lambda) \frac{K_{1+\lambda}(\zeta)}{K_\lambda(\zeta)} - \zeta \left[ \frac{K_{1+\lambda}(\zeta)}{K_\lambda(\zeta)} \right]^2 \right\} \\ &= \frac{\delta^2}{\zeta} r + \frac{\beta^2 \delta^4}{\zeta^3} [\zeta + 2(1+\lambda)r - \zeta r^2]. \end{aligned} \quad (80)$$

Usually, the asymmetry of the distribution is small implying that  $c_1$  and thus  $\beta$  is small. It is then useful to introduce the symmetric variance

$$\sigma_s^2 \equiv \delta^2 r / \zeta, \quad (81)$$

and write

$$c_1 = \beta \sigma_s, \quad c_2 = \sigma^2 = \sigma_s^2 + \beta^2 \left[ \frac{\delta^4}{\zeta^2} + 2(1+\lambda) \frac{\delta^2}{\zeta^2} \sigma_s - \sigma_s^2 \right] \quad (82)$$

The cumulants  $c_3$  and  $c_4$  are most compactly written as

$$\begin{aligned} c_3 &= \beta \left[ \frac{3\delta^4}{\zeta^2} + 6(1+\lambda) \frac{\delta^2}{\zeta^2} \sigma_s^2 - 3\sigma_s^4 \right] \\ &+ \beta^3 \left\{ 2(2+\lambda) \frac{\delta^6}{\zeta^4} + [4(1+\lambda)(2+\lambda) - 2\zeta^2] \frac{\delta^4}{\zeta^4} \sigma_s^2 - 6(1+\lambda) \frac{\delta^2}{\zeta^2} \sigma_s^4 + 2\sigma_s^6 \right\} \end{aligned} \quad (83)$$

and

$$\begin{aligned} c_4 &= \kappa \sigma^4 = \frac{3\delta^4}{\zeta^2} + \frac{6\delta^2}{\zeta^2} (1+\lambda) \sigma_s^2 - 3\sigma_s^4 \\ &+ 6\beta^2 \left\{ 2(2+\lambda) \frac{\delta^6}{\zeta^4} + [4(1+\lambda)(2+\lambda) - 2\zeta^2] \frac{\delta^4}{\zeta^4} \sigma_s^2 - 6(1+\lambda) \frac{\delta^2}{\zeta^2} \sigma_s^4 + 2\sigma_s^6 \right\} \\ &+ \beta^4 \left\{ [4(2+\lambda)(3+\lambda) - \zeta^2] \frac{\delta^8}{\zeta^6} + [4(1+\lambda)(2+\lambda)(3+\lambda) - 2(5+4\lambda)\zeta^2] \frac{\delta^6}{\zeta^6} \sigma_s^2 \right. \\ &\quad \left. - 2[(1+\lambda)(11+7\lambda) - 2\zeta^2] \frac{\delta^4}{\zeta^4} \sigma_s^4 + 12(1+\lambda) \frac{\delta^2}{\zeta^2} \sigma_s^6 - 3\sigma_s^8 \right\}. \end{aligned} \quad (84)$$

The first term in  $c_4$  is equal to  $\sigma_s^4$  times the kurtosis of the symmetric distribution

$$\kappa_s \equiv \frac{3\delta^4}{\zeta^2 \sigma_s^4} + \frac{6\delta^2}{\zeta^2 \sigma_s^2} (1+\lambda) - 3, \quad (85)$$

Inserting here  $\sigma_s^2$  from (81), we find

$$\kappa_s \equiv \frac{3}{r^2(\zeta)} + (1 + \lambda) \frac{6}{\zeta r(\zeta)} - 3. \quad (86)$$

Since all Bessel functions  $K_\nu(z)$  have the same large- $z$  behavior  $K_\nu(z) \rightarrow \sqrt{\pi/2z} e^{-z}$  and the small- $z$  behavior  $K_\nu(z) \rightarrow \Gamma(\nu)/2(z/2)^\nu$ , the kurtosis starts out at  $3/\lambda$  for  $\zeta = 0$  and decreases monotonously to 0 for  $\zeta \rightarrow \infty$ . Thus a high kurtosis can be reached only with a small parameter  $\lambda$ .

The first term in  $c_3$  is  $\beta \kappa_s \sigma_s^4$ , and the first two terms in  $c_4$  are  $\kappa_s \sigma_s^4 + 6(c_3/\beta - \kappa_s \sigma_s^4)$ . For a symmetric distribution with certain variance  $\sigma_s^2$  and kurtosis  $\kappa_s$  we select some parameter  $\lambda < 3/\kappa_s$ , and solve the Eq. (86) to find  $\zeta$ . This is inserted into Eq. (81) to determine

$$\delta^2 = \frac{\sigma_s^2 \zeta}{r(\zeta)}. \quad (87)$$

For larger kurtosis, the kurtosis is not an optimal parameter to determine generalized hyperbolic distributions. A better fit to the data is reached by reproducing correctly the size and shape of the near the peak and allow for some deviations in the tails of the distribution, on which the kurtosis depends quite sensitively.

For distributions which are only slightly asymmetric, which is usually the case, it is sufficient to solve the above symmetric equations and determine the small parameter  $\beta$  approximately by the skew  $s = c_3/\sigma^3$  from the first line in (83) as

$$\beta \approx \frac{s}{\kappa_s \sigma_s}. \quad (88)$$

This approximation can be improved iteratively by reinserting  $\beta$  into the second equation in (82) to determine from the variance of the data  $\sigma^2$  an improved value of  $\sigma_s^2$ , then into the first two lines of Eq. (84) to determine from the kurtosis  $\kappa$  of the data an improved value of  $\kappa_s$ , and so on.

### E. Debye-Waller Factor for Non-Gaussian Fluctuations

An important quantity for fluctuating Gaussian variables is the . It was introduced in solid state physics to describe the reduction of intensity of Bragg peaks due to the thermal fluctuations of the atomic positions. If  $\mathbf{u}(\mathbf{x})$  is the atomic displacement field, the Gaussian approximation to the Debye-Waller factor  $e^{-2W}$  is given by

$$e^{-W} \equiv \langle e^{-\nabla \cdot \mathbf{u}(\mathbf{x})} \rangle = e^{-\Sigma_{\mathbf{k}} \langle |\mathbf{k} \cdot \mathbf{u}(\mathbf{k})|^2 \rangle / 2}, \quad (89)$$

This is a direct consequence of the relation for Gauss distributions

$$\langle e^{Px} \rangle \equiv \int dx \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} e^{Px} = \int dx \int \frac{dp}{2\pi} e^{-\sigma^2 p^2/2} e^{ipx+Px} = e^{\sigma^2 P^2/2}, \quad (90)$$

which is a manifestation of Wick's rule

$$\langle e^{Px} \rangle = e^{P^2 \langle x^2 \rangle / 2}, \quad (91)$$

whose right-hand side may be considered as the Debye-Waller factor for the Gaussian variable  $x$ .

There exists a simple generalization of this relation to non-Gaussian distributions, which is

$$\langle e^{Px} \rangle \equiv \int dx \tilde{L}_{\sigma^2}^{(\mu, \alpha)}(x) e^{Px} = \int dx \int \frac{dp}{2\pi} e^{-H(p)} e^{ipx+Px} = e^{-H(iP)}. \quad (92)$$

## F. Path Integral for Non-Gaussian Distribution

Let us calculate the properties of the simplest process whose fluctuations are distributed according to any of the general non-Gaussian distributions. We consider the stochastic differential equation

$$\dot{x}(t) = r_x + \eta(t), \quad (93)$$

where the noise variable  $\eta(t)$  is distributed according an arbitrary distribution. The constant growth rate  $r_x$  in (93) is uniquely defined only if the average of the noise variable vanishes:  $\langle \eta(t) \rangle = 0$ . The general distributions discussed above can have a nonzero average  $\langle x \rangle = c_1$  which has to be subtracted from  $\eta(t)$  to identify  $r_x$ . The subsequent discussion will be simplest if we imagine  $r_x$  to have replaced  $c_1$  in the above distributions, i.e., that the power series expansion of the Hamiltonian (49) is replaces as follows:

$$\begin{aligned} H(p) &\rightarrow H_{r_x}(p) \equiv H(p) - H'(0)p + ir_x p \equiv \tilde{H}(p) + ir_x p \\ &\equiv ir_x p + \frac{1}{2}c_2 p^2 - i\frac{1}{3!}c_3 p^3 - \frac{1}{4!}c_4 p^4 + i\frac{1}{5!}c_5 p^5 + \dots \end{aligned} \quad (94)$$

Thus we may simply work with the original expansion (49) and replace, at the end,

$$c_1 \rightarrow r_x. \quad (95)$$

The stochastic differential equation (93) can be assumed to read simply

$$\dot{x}(t) = \eta(t), \quad (96)$$

With the ultimate replacement (95) in mind, the probability distribution of the endpoints for the paths starting at a certain initial point is given by a path integral of the form

$$P(x_b t_b | x_a t_a) = \int \mathcal{D}\eta \int \mathcal{D}x \exp \left[ - \int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] \delta[\dot{x} - \eta], \quad (97)$$

with the initial condition  $x(t_a) = x_a$ . The final point is, of course,  $x_b = x(t_b)$ .

The function  $\tilde{H}(\eta)$  to be modified at the end by the replacement (94) is the negative logarithm of the truncated L  vy distribution or the generalized hyperbolic distribution or any other distribution. By analogy with the definition in momentum space (21) we shall define  $\tilde{H}(\eta)$  by

$$e^{-\tilde{H}(x)} \equiv \tilde{L}_{\sigma^2}^{(\lambda, \alpha)}(x), \quad \text{or} \quad e^{-\tilde{H}(x)} \equiv \tilde{G}(x), \quad (98)$$

or for any distribution function  $\tilde{D}(x)$  generically by

$$e^{-\tilde{H}(x)} \equiv \tilde{D}(x). \quad (99)$$

The correlation functions of the noise variable  $\eta(t)$  in the path integral (97) are given by a straightforward functional generalization of formulas (53). For this purpose, we express the noise distribution  $P[\eta] \equiv \exp \left[ - \int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right]$  in (97) as a Fourier path integral

$$P[\eta] = \int \mathcal{D}\eta \int \frac{\mathcal{D}p}{2\pi} \exp \left\{ \int_{t_a}^{t_b} dt [ip(t)\eta(t) - H(p(t))] \right\}, \quad (100)$$

and note that the correlation functions can be obtained from the functional derivatives

$$\langle \eta(t_1) \cdots \eta(t_n) \rangle = (-i)^n \int \mathcal{D}\eta \int \frac{\mathcal{D}p}{2\pi} \left[ \frac{\delta}{\delta p(t_1)} \cdots \frac{\delta}{\delta p(t_n)} e^{i \int_{t_a}^{t_b} dt p(t)\eta(t)} \right] e^{- \int_{t_a}^{t_b} dt H(p(t))}.$$

After  $n$  partial integrations, this becomes

$$\begin{aligned} \langle \eta(t_1) \cdots \eta(t_n) \rangle &= i^n \int \mathcal{D}\eta \int \frac{\mathcal{D}p}{2\pi} e^{i \int_{t_a}^{t_b} dt p(t)\eta(t)} \frac{\delta}{\delta p(t_1)} \cdots \frac{\delta}{\delta p(t_n)} e^{- \int_{t_a}^{t_b} dt H(p(t))} \\ &= i^n \left[ \frac{\delta}{\delta p(t_1)} \cdots \frac{\delta}{\delta p(t_n)} e^{- \int_{t_a}^{t_b} dt H(p(t))} \right]_{p(t) \equiv 0}. \end{aligned} \quad (101)$$

By expanding the exponential  $e^{-\int_{t_a}^{t_b} dt H(p(t))}$  in a power series (49) we find immediately

$$\langle \eta(t_1) \rangle \equiv Z^{-1} \int \mathcal{D}\eta \eta(t_1) \exp \left[ - \int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] = c_1, \quad (102)$$

$$\begin{aligned} \langle \eta(t_1) \eta(t_2) \rangle &\equiv Z^{-1} \int \mathcal{D}\eta \eta(t_1) \eta(t_2) \exp \left[ - \int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] \\ &= c_2 \delta(t_1 - t_2) + c_1^2, \end{aligned} \quad (103)$$

$$\begin{aligned} \langle \eta(t_1) \eta(t_2) \eta(t_3) \rangle &\equiv Z^{-1} \int \mathcal{D}\eta \eta(t_1) \eta(t_2) \eta(t_3) \exp \left[ - \int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] \\ &= c_3 \delta(t_1 - t_2) \delta(t_1 - t_3) \\ &\quad + c_2 c_1 [\delta(t_1 - t_2) + \delta(t_2 - t_3) + \delta(t_1 - t_3)] + c_1^3, \end{aligned} \quad (104)$$

$$\begin{aligned} \langle \eta(t_1) \eta(t_2) \eta(t_3) \eta(t_4) \rangle &\equiv Z^{-1} \int \mathcal{D}\eta \eta(t_1) \eta(t_2) \eta(t_3) \eta(t_4) \exp \left[ - \int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] \\ &= c_4 \delta(t_1 - t_2) \delta(t_1 - t_3) \delta(t_1 - t_4) \\ &\quad + c_3 c_1 [\delta(t_1 - t_2) \delta(t_1 - t_3) + 3 \text{ cyclic perms}] \\ &\quad + c_2^2 [\delta(t_1 - t_2) \delta(t_3 - t_4) + \delta(t_1 - t_3) \delta(t_2 - t_4) + \delta(t_1 - t_4) \delta(t_2 - t_3)] \\ &\quad + c_2 c_1^2 [\delta(t_1 - t_2) + 5 \text{ pair terms}] + c_1^4, \end{aligned} \quad (105)$$

where

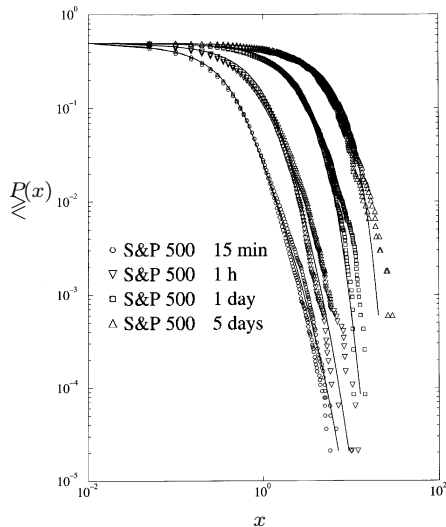
$$Z \equiv \int \mathcal{D}\eta \exp \left[ - \int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right]. \quad (106)$$

The higher correlation functions are obvious generalizations of (39). The different contributions on the right-hand side of (103)–(105) Eq. (105) are distinguishable by their connectedness structure.

An important property of the probability (97) is that it satisfies the semigroup property

$$P(x_c t_c | x_a t_a) = \int_{-\infty}^{\infty} dx_b P(x_c t_c | x_b t_b) P(x_b t_b | x_a t_a). \quad (107)$$

In Fig. 10 we show that this property is satisfied by experimental asset distributions reasonably well, except in the low-probability tails. The discrepancy manifests itself also at another place: We shall see in Subsection III G that the solution of the path integral has a kurtosis decreasing inversely proportional to the time. The data in Fig. 10, however, show only an slower inverse square-root falloff. This can be accounted for in the theory by including fluctuations of the width  $\sigma$ , which are certainly present as was illustrated before in Figs. 3 and 4. For calculations of this type with Gaussian distributions see Ref. [34]. If the semigroup property was satisfied perfectly, the Lévy parameter  $\alpha$  would be time independent as we can see from Eq. (42) with  $\sigma^2 \propto (t_b - t_a)$  and  $\kappa \propto 1/(t_b - t_a)$ . With the slower falloff of  $\kappa \propto 1/\sqrt{(t_b - t_a)}$ , however,  $\alpha$  decreases like  $1/\sqrt{(t_b - t_a)}$ .



$$\kappa \approx \text{const.} \times 1/(t_b - t_a)^{0.5}$$

see extra fig6b.pdf

$(t_b - t_a)/5 \text{ min}$



FIG. 10. Left-hand side: Cumulative distributions obtained from repeated convolution integrals of the 15-min distribution (from Ref. [5]). Apart from the far ends of the tails, the semigroup property (107) is reasonably well satisfied. Right-hand side: Falloff of kurtosis is slower than  $1/(t_b - t_a)$  expected from convolution property .

### G. Fokker-Planck-Type Equation

The  $\delta$ -functional may be represented by a Fourier integral leading to

$$P(x_b t_b | x_a t_a) = \int \mathcal{D}\eta \int \frac{\mathcal{D}p}{2\pi} \exp \left\{ \int_{t_a}^{t_b} dt \left[ ip(t)\dot{x}(t) - ip(t)\eta(t) - \tilde{H}(\eta(t)) \right] \right\}. \quad (108)$$

Integrating out the noise variable  $\eta(t)$  amounts to performing the inverse Fourier transform (20) at each instant of time and we obtain

$$P(x_b t_b | x_a t_a) = \int \frac{\mathcal{D}p}{2\pi} \exp \left\{ \int_{t_a}^{t_b} dt \left[ ip(t)\dot{x}(t) - H(p(t)) \right] \right\}. \quad (109)$$

Integrating over all  $x(t)$  with fixed end points enforces a constant momentum along the path, and we remain with a single integral

$$P(x_b t_b | x_a t_a) = \int \frac{dp}{2\pi} \exp \left[ -(t_b - t_a)H(p) + ip(x_b - x_a) \right]. \quad (110)$$

The Fourier integral can now be performed and we obtain, for a truncated Lévy distribution:

$$P(x_b t_b | x_a t_a) = \tilde{L}_{\sigma^2(t_b - t_a)}^{(\mu, \alpha)}(x_b - x_a). \quad (111)$$

The result is therefore a truncated Lévy distribution of increasing width. All expansion coefficients  $c_n$  of  $H(p)$  in Eq. (35) receive the same factor  $t_b - t_a$ , which has the consequence that the kurtosis  $\kappa = c_4/c_2^2$  decreases inversely proportional to  $t_b - t_a$ . The distribution becomes increasingly Gaussian with increasing time, as a manifestation of the central limiting theorem of statistical mechanics. This is in contrast to the pure Lévy distribution which has no finite width and therefore maintains its power falloff at large distances.

From the Fourier representation (110) it is easy to prove that this probability satisfies a Fokker-Planck-type equation

$$\partial_t P(x_b t_b | x_a t_a) = -H(-i\partial_x)P(x_b t_b | x_a t_a). \quad (112)$$

Indeed, the general solution  $\psi(x, t)$  of this differential equation with the initial condition  $\psi(x, 0)$  is given by the path integral generalizing (97)

$$\psi(x, t) = \int \mathcal{D}\eta \exp \left[ - \int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] \psi \left( x - \int_{t_a}^t dt' \eta(t') \right). \quad (113)$$

To verify that this satisfies indeed the Fokker-Planck-type equation (112) we consider  $\psi(x, t)$  at a slightly later time  $t + \epsilon$  and expand

$$\begin{aligned} \psi(x, t + \epsilon) &= \int \mathcal{D}\eta \exp \left[ - \int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] \psi \left( x - \int_{t_a}^t dt' \eta(t') - \int_t^{t+\epsilon} dt' \eta(t') \right) \\ &= \int \mathcal{D}\eta \exp \left[ - \int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] \left\{ \psi \left( x - \int_{t_a}^t dt' \eta(t') \right) \right. \\ &\quad - \psi' \left( x - \int_{t_a}^t dt' \eta(t') \right) \int_t^{t+\epsilon} dt' \eta(t') \\ &\quad + \frac{1}{2} \psi'' \left( x - \int_{t_a}^t dt' \eta(t') \right) \int_t^{t+\epsilon} dt_1 dt_2 \eta(t_1) \eta(t_2) \\ &\quad - \frac{1}{3!} \psi''' \left( x - \int_{t_a}^t dt' \eta(t') \right) \int_t^{t+\epsilon} dt_1 dt_2 dt_3 \eta(t_1) \eta(t_2) \eta(t_3) \\ &\quad \left. + \frac{1}{4!} \psi^{(4)} \left( x - \int_{t_a}^t dt' \eta(t') \right) \int_t^{t+\epsilon} dt_1 dt_2 dt_3 dt_4 \eta(t_1) \eta(t_2) \eta(t_3) \eta(t_4) + \dots \right\}. \end{aligned} \quad (114)$$

Inserting here correlation functions (102)–(105) we obtain

$$\begin{aligned} \psi(x, t + \epsilon) &= \int \mathcal{D}\eta \exp \left[ - \int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] \\ &\times \left[ -\epsilon c_1 \partial_x + (\epsilon c_2 + \epsilon^2 c_1) \frac{1}{2} \partial_x^2 - (\epsilon c_3 + 3\epsilon^2 c_2 c_1) \frac{1}{3!} \partial_x^3 \right. \\ &\left. + (\epsilon c_4 + \epsilon^2 4c_3 c_1 + \epsilon^2 3c_2^2 + \epsilon^3 c_2 c_1^2 + \epsilon^4 c_1^3) \frac{1}{4!} \partial_x^4 + \dots \right] \psi \left( x - \int_{t_a}^t dt' \eta(t') \right). \end{aligned} \quad (115)$$

In the limit  $\epsilon \rightarrow 0$ , only the linear terms in  $\epsilon$  contribute, which are all due to the connected parts of the correlation functions of  $\eta(t)$ . The differential operators in the brackets can now be pulled out of the integral and we find the differential equation

$$\partial_t \psi(x, t) = \left[ -c_1 \partial_x + c_2 \frac{1}{2} \partial_x^2 - c_3 \frac{1}{3!} \partial_x^3 + c_4 \frac{1}{4!} \partial_x^4 + \dots \right] \psi(x, t). \quad (116)$$

We now replace  $c_1 \rightarrow r_x$  and express using (94) the differential operators in brackets as Hamiltonian operator  $-H_{r_x}(-i\partial_x)$ . This leads to the Schrödinger-like equation

$$\partial_t \psi(x, t) = -H_{r_x}(-i\partial_x) \psi(x, t). \quad (117)$$

Note that due to the many derivatives in  $H(i\partial_x)$ , this equation is in general non-local.

By a similar procedure as in the derivation of Eq. (116) it is possible to derive a generalization of Ito's rule (6) to functions of noise variable with non-Gaussian distributions. Thus we expand  $f(x(t + \epsilon))$  as in (7) where  $x(t)$  satisfies now the stochastic differential equation  $\dot{x}(t) = \eta(t)$  with a nonzero expectation value  $\langle \eta(t) \rangle = c_1$ . In contrast to the previous evaluation of (7) with Gaussian noise, which had to be carried out only up to second order in the noise variable, we must now keep *all* orders. Evaluating the noise averages of the multiple integrals on the right-hand side using the correlation functions (102)–(105), we find the time derivative of the expectation value of an arbitrary function of the fluctuating variable  $x(t)$

$$\begin{aligned} \langle \dot{f}(x(t)) \rangle &= \langle f(x(t)) \rangle + \langle f'(x(t)) \rangle \epsilon c_1 + \frac{1}{2} \langle f''(x(t)) \rangle (\epsilon c_2 + \epsilon^2 c_1^2) \\ &\quad + \frac{1}{3!} \langle f^{(3)}(x(t)) \rangle (\epsilon c_3 + \epsilon^2 c_2 c_1 + \epsilon^3 c_1^3) + \dots \\ &= \epsilon \left[ -c_1 \partial_x + c_2 \frac{1}{2} \partial_x^2 - c_3 \frac{1}{3!} \partial_x^3 + \dots \right] \langle f(x(t)) \rangle + \mathcal{O}(\epsilon^2). \end{aligned} \quad (118)$$

After the replacement  $c_1 \rightarrow r_x$  the function  $f(x(t))$  obeys therefore the following equation:

$$\langle \dot{f}(x(t)) \rangle = -H_{r_x}(i\partial_x) \langle f(x(t)) \rangle. \quad (119)$$

Taking out the lowest-derivative term this takes a form

$$\langle \dot{f}(x(t)) \rangle = \langle \partial_x f(x(t)) \rangle \langle \dot{x}(t) \rangle - \bar{H}_{r_x}(i\partial_x) \langle f(x(t)) \rangle. \quad (120)$$

In postpoint time slicing, this may be viewed as the expectation value of the stochastic differential equation

$$\dot{f}(x(t)) = \partial_x f(x(t)) \dot{x}(t) - \bar{H}_{r_x}(i\partial_x) f(x(t)). \quad (121)$$

This is the direct generalization of Ito's rule (6).

For an exponential function  $f(x) = e^{Px}$ , this becomes

$$\frac{d}{dt} e^{Px(t)} = [\dot{x}(t) - H_{r_x}(iP)] e^{Px(t)}. \quad (122)$$

As a consequence of this equation for  $P = 1$ , the rate  $r_S$  with which a stock price  $S(t) = e^{x(t)}$  grows on the average according to formula (1) is now related to  $r_x$  by

$$r_S = r_x - \bar{H}(i) = r_x - [H(i) - iH'(0)] = -H_{r_x}(i), \quad (123)$$

which replaces the simple Ito relation  $r_S = r_x + \sigma^2/2$  in Eq. (10). Recall the definition  $\bar{H}(p) \equiv H(p) - H'(0)p$  in Eq. (94). The corresponding generalization of the left-hand part of Eq. (5) reads

$$\frac{\dot{S}}{S} = \dot{x}(t) - \bar{H}(i) = \dot{x}(t) - [H(i) - iH'(0)] = \dot{x}(t) - r_x - H_{r_x}(i). \quad (124)$$

The forward price of a stock must therefore be calculated with the generalization of formula (11):

$$\langle S(t) \rangle = S(0)e^{r_S t} = S(0) \langle e^{r_x t + \int_0^t dt' \eta(t')} \rangle = S(0)e^{-H_{r_x}(i)t} = S(0)e^{\{r_x - [H(i) - iH'(0)]t\}}. \quad (125)$$

Note that one may derive the differential equation of an arbitrary function  $f(x(t))$  in Eq. (119) from a simple mnemonic rule, expanding sloppily

$$\begin{aligned} f(x(t+dt)) &= f(x(t) + \dot{x}dt) = f(x(t)) + f'(x(t))\dot{x}(t)dt + \frac{1}{2}f''(x(t))\dot{x}^2(t)dt^2 \\ &+ \frac{1}{3!}f^{(3)}(x(t))\dot{x}^3(t)dt^3 + \dots, \end{aligned} \quad (126)$$

and replacing

$$\langle \dot{x}(t) \rangle dt \rightarrow c_1 dt, \quad \langle \dot{x}^2(t) \rangle dt^2 \rightarrow c_2 dt, \quad \langle \dot{x}^3(t) \rangle dt^2 \rightarrow c_3 dt, \dots \quad (127)$$

#### IV. CONCLUSION

The new stochastic calculus developed in this paper should be useful for estimating financial risks of a variety of investments. In particular, it will help in developing simple methods to estimating fair option prices for stocks with non-Gaussian fluctuations. More details can be found in the textbook Ref. [38].

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